

ENUMERATIONS OF VERTICES AMONG ALL ROOTED ORDERED TREES WITH LEVELS AND DEGREES

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ABSTRACT. In this paper we enumerate and give bijections for the following four sets of vertices among rooted ordered trees of a fixed size: (i) first-children of degree k at level ℓ , (ii) non-first-children of degree k at level $\ell - 1$, (iii) leaves having $k - 1$ elder siblings at level ℓ , and (iv) non-leaves of outdegree k at level $\ell - 1$. Our results unite and generalize several previous works in the literature.

1. INTRODUCTION

Let \mathcal{T}_n be the set of *rooted ordered trees* with n edges. It is well known that the cardinality of \mathcal{T}_n is the n -th *Catalan number*

$$\text{Cat}_n = \frac{1}{n+1} \binom{2n}{n}.$$

For example, there are 5 rooted ordered trees with 3 edges, see Figure 1. Clearly the number of vertices among trees in \mathcal{T}_n is

$$(n+1) \text{Cat}_n = \binom{2n}{n}.$$

Given a rooted ordered tree, a vertex v is a *child* of a vertex u and u is the *parent* of v if v is directly connected to u when moving away from the root. A vertex without children is called a *leaf*. Note that by definition the root is not a child. Vertices with the same parent are called *siblings*. Since siblings are linearly ordered, when drawing trees the siblings are put in the left-to-right order. Siblings to the left of v are called an *elder* siblings of v . The leftmost vertex among siblings is called the *first-child*. In Figure 1, there are 10 first-children as well as 10 leaves, which is precisely a half of all 20 vertices in trees in \mathcal{T}_3 .

In 1999, Shapiro [Sha99] proved the following using generating functions.

Theorem 1. *For any positive integer n , the following four sets of vertices among trees in \mathcal{T}_n are equinumerous:*

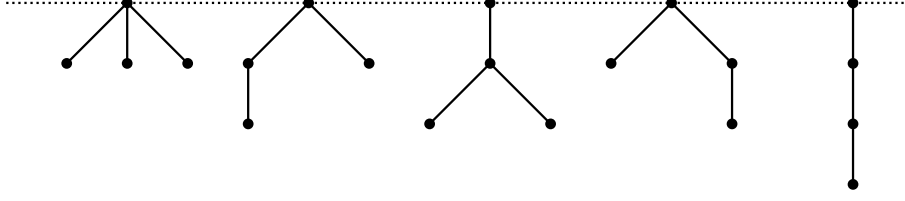
- (i) *first-children,*
- (ii) *non-first-children,*
- (iii) *leaves, and*
- (iv) *non-leaves.*

Here, the cardinality of each set is

$$\frac{1}{2} \binom{2n}{n},$$

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FIGURE 1. The five trees in \mathcal{T}_3

which is a half of the number of vertices among trees in \mathcal{T}_n .

Seo and Shin [SS02] gave an involution proving (iii) and (iv) are equinumerous. The equality with the other two sets is easily seen: (i) and (iv) are equinumerous since each non-leaf has its unique first-child and the union of (i) and (ii) are the same with (iii) and (iv).

1.1. Degree and outdegree. The *degree* of a vertex is the number of edges incident to it. As every edge has a natural outward direction away from the root, we can have the notion of the *outdegree* of a vertex v , which is the number of edges starting at v and pointing away from the root. Since each vertex has a degree and each non-leaf has a positive outdegree, Theorem 1 can be restated as follows.

Theorem 1' (Shapiro, 1999). *Let $n \geq 1$. Among trees in \mathcal{T}_n , the number of vertices of positive degree equals twice the number of vertices of positive outdegree.*

In 2004, Eu, Liu, and Yeh [ELY04] proved combinatorially the following by constructing a two-to-one correspondence which answered a problem posed by Deutsch and Shapiro [DS01, p. 259].

Theorem 2 (Eu, Liu, and Yeh, 2004). *Let $n \geq 1$. Among trees in \mathcal{T}_n , the number of vertices of odd degree equals twice the number of vertices of odd positive outdegree.*

In light of these two results, it is natural to ask if more can be said. In Corollary 6 we will prove that for any positive integer k the number of vertices of degree k always equals *twice* the number of that of outdegree k among trees in \mathcal{T}_n .

1.2. Even and odd level. A vertex v in a rooted tree T is at *level* ℓ if the distance (number of edges) from the root to v is ℓ . By an involution, the following result was obtained by Chen, Li, and Shapiro [CLS07].

Theorem 3 (Chen, Li, and Shapiro, 2007). *The number of vertices at odd levels equals the number of vertices at even levels among trees in \mathcal{T}_n .*

It is again natural to ask if more can be said. In Corollary 8 of this paper we will prove that for any positive integer k the number of vertices of degree k at odd levels always equals the number of vertices of degree k at even levels among trees in \mathcal{T}_n .

1.3. Main result. In this paper we generalize the above by regarding both degrees and levels simultaneously. We will give a combinatorial proof for the following main result.

Theorem 4. *Given $n \geq 1$, for any two positive integers k and ℓ , there are one-to-one correspondences between the following four sets of vertices among trees in \mathcal{T}_n :*

- (i) *first-children of degree k at level ℓ ,*
- (ii) *non-first-children of degree k at level $(\ell - 1)$,*
- (iii) *leaves having exactly $(k - 1)$ elder siblings at level ℓ , and*
- (iv) *non-leaves of outdegree k at level $(\ell - 1)$.*

Here, the cardinality of each set is

$$\frac{k + 2\ell - 2}{2n - k} \binom{2n - k}{n + \ell - 1}. \quad (1)$$

The rest of the paper is organized as follows. In Section 2 we derive several corollaries from Theorem 4, including generalizations of Theorem 1 through 3. The proof of Theorem 4 is given in the next two sections: In Section 3 we construct bijections between these four sets while in Section 4 we show combinatorially that the cardinality is given by (1).

2. COROLLARIES

In 2012, Cheon and Shapiro [CS12, Example 2.2] gave a formula for the number of vertices of outdegree k by generating function arguments. Summing over $\ell \geq 1$ in Theorem 4 yields the following, in which the fourth item recovers bijectively the result of Cheon and Shapiro.

Corollary 5. *Given $n \geq 1$ and $k \geq 1$, there are one-to-one correspondences between the following four sets of vertices among trees in \mathcal{T}_n :*

- (i) *first-children of degree k ,*
- (ii) *non-first-children of degree k ,*
- (iii) *leaves having exactly $(k - 1)$ elder siblings, and*
- (iv) *non-leaves of outdegree k .*

Here, the cardinality of each set is

$$\binom{2n - k - 1}{n - 1}.$$

Proof. The result follows by a telescoping summation on $\ell \geq 1$ using the formula

$$\frac{k + 2\ell - 2}{2n - k} \binom{2n - k}{n + \ell - 1} = \binom{2n - k - 1}{n + \ell - 2} - \binom{2n - k - 1}{n + \ell - 1}.$$

□

The result mentioned in Subsection 1.1 can be obtained from (i), (ii), and (iv) of Corollary 5.

Corollary 6. *Given $n \geq 1$ and $k \geq 1$, the number of vertices of degree k equals twice the number of vertices of outdegree k among trees in \mathcal{T}_n .*

Note that Corollary 6 is a refinement of Theorem 2. Summing over all k in Theorem 4 yields the following.

Corollary 7. *Given $n \geq 1$ and $\ell \geq 1$, there are one-to-one correspondences between the following four sets of vertices among trees in \mathcal{T}_n :*

- (i) *first-children at level ℓ ,*
- (ii) *non-first-children at level $(\ell - 1)$,*
- (iii) *leaves at level ℓ , and*
- (iv) *non-leaves at level $(\ell - 1)$.*

Here, the cardinality of each set is

$$\frac{\ell}{n} \binom{2n}{n+\ell}.$$

Proof. The result follows by a telescoping summation on $k \geq 1$ using the following formula,

$$\frac{k+2\ell-2}{2n-k} \binom{2n-k}{n+\ell-1} = \frac{k+2\ell-1}{2n-k+1} \binom{2n-k+1}{n+\ell} - \frac{k+2\ell}{2n-k} \binom{2n-k}{n+\ell}.$$

□

The result mentioned in Subsection 1.2 can be obtained from (i) and (ii) in Theorem 4.

Corollary 8. *Given $n \geq 1$, the number of vertices of degree k at odd levels equals the number of vertices of degree k at even levels among trees in \mathcal{T}_n .*

3. PROOF OF THEOREM 4: BIJECTIONS

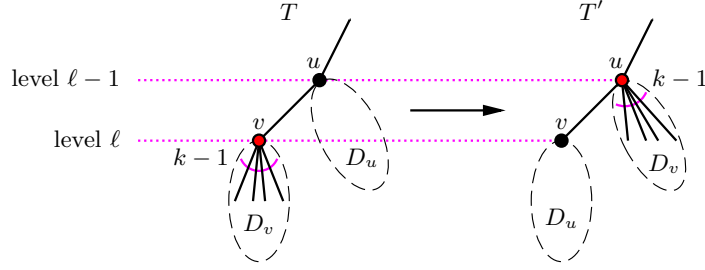
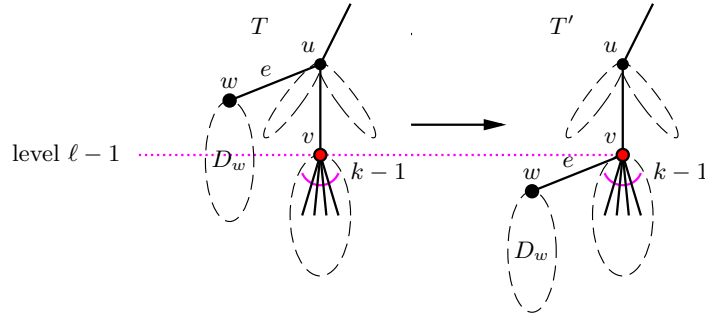
Denote \mathcal{TV}_n by the set of pairs (T, v) satisfying $T \in \mathcal{T}_n$ and $v \in V(T)$. Given positive integers n , k , and ℓ , we let

- (i) \mathcal{A} denote the set of $(T, v) \in \mathcal{TV}_n$ such that v is a first-child of degree k at level ℓ in T ,
- (ii) \mathcal{B} denote the set of $(T, v) \in \mathcal{TV}_n$ such that v is a non-first-child of degree k at level $(\ell - 1)$ in T ,
- (iii) \mathcal{C} denote the set of $(T, v) \in \mathcal{TV}_n$ such that v is a leaf having exactly $(k - 1)$ elder siblings at level ℓ in T , and
- (iv) \mathcal{D} denote the set of $(T, v) \in \mathcal{TV}_n$ such that v is a non-leaf of outdegree k at level $(\ell - 1)$ in T .

(i) \Leftrightarrow (iv). A bijection from \mathcal{A} to \mathcal{D} is constructed as follows: given $(T, v) \in \mathcal{A}$, find the parent vertex u of v . As the Figure 2, consider a subtree D_v of v and another subtree D_u of u on the right of the edge (u, v) . By interchanging D_v and D_u we get the new tree T' with the vertex u such that

$$\text{outdeg}(T', u) = \deg(T, v) = k, \quad \text{lev}(T', u) = \text{lev}(T, v) - 1 = \ell - 1,$$

where $\text{lev}(T, v)$ (resp. $\deg(T, v)$, $\text{outdeg}(T, v)$) means the level (resp. degree, outdegree) of a vertex v in a tree T . Thus, (T', u) belongs to \mathcal{D} . Since this interchanging action is reversible, it is a one-to-one correspondence.


 FIGURE 2. A bijection from \mathcal{A} to \mathcal{D}

 FIGURE 3. A bijection from \mathcal{B} to \mathcal{D} if v is not the root of T

(ii) \Leftrightarrow (iv). A bijection from \mathcal{B} to \mathcal{D} is constructed as follow: given $(T, v) \in \mathcal{B}$, we perform the following action:

(a) if v is the root of T , the pair (T, v) also belongs to \mathcal{D} due to

$$\text{outdeg}(T, v) = \deg(T, v) = k.$$

(b) if v is not the root of T , find the parent vertex u of v , the first-child w of u and the edge $e = (u, w)$. As the Figure 3, cut and paste the edge e and the subtree D_w consisting of all descendants of w from u to v such that the vertex w is the first-child of v . We obtain the tree T' and the vertex v in T' satisfies

$$\text{outdeg}(T', v) = \deg(T, v) = k, \quad \text{lev}(T', v) = \text{lev}(T, v) = \ell - 1.$$

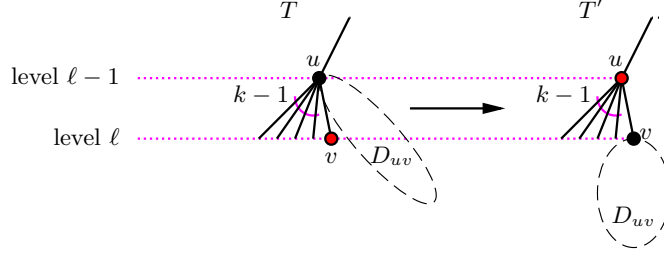
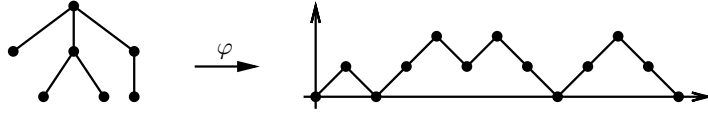
Thus, (T', v) belongs to \mathcal{D} .

Since this action is reversible, it is a one-to-one correspondence.

(iii) \Leftrightarrow (iv). A bijection from \mathcal{C} to \mathcal{D} is constructed as follows: given $(T, v) \in \mathcal{C}$, find the parent vertex u of v . As the Figure 4, consider the subtree D_{uv} of u on the right of the edge (u, v) . By cutting and pasting the subtrees D_{uv} from u to v we get the new tree T' with the vertex u such that

$$\text{outdeg}(T', u) = \text{eld}(T, v) + 1 = k, \quad \text{lev}(T', u) = \text{lev}(T, v) - 1 = \ell - 1,$$

where $\text{eld}(T, v)$ means the number of elder siblings of a vertex v in a tree T . Thus, (T', u) belongs to \mathcal{D} . Since this interchanging action is reversible, it is a one-to-one correspondence.

FIGURE 4. A bijection from \mathcal{C} to \mathcal{D} FIGURE 5. The bijection φ

4. PROOF OF THEOREM 4: ENUMERATION

By putting $\ell + 1$ in place of ℓ in (iv) of Theorem 4, it suffices to show that for any two nonnegative integers k and ℓ , the number of vertices of outdegree k at level ℓ among trees in \mathcal{T}_n is

$$\frac{k + 2\ell}{2n - k} \binom{2n - k}{n + \ell}.$$

The following lemma gives a cumulative counting in k and ℓ .

Lemma 9 (Main lemma). *Given $n \geq 1$, for any two nonnegative integers k and ℓ , the number of vertices of outdegree at least k and at level at least ℓ among trees in \mathcal{T}_n is*

$$\binom{2n - k}{n + \ell}. \quad (2)$$

Proof. Let \mathcal{V} be the set of $(T, v) \in \mathcal{TV}_n$ such that v is a vertex of outdegree at least k and at level at least ℓ in T . Let \mathcal{L} be the set of *lattice paths* of length $(2n - k)$ from (k, k) to $(2n, -2\ell)$, consisting of $(n - k - \ell)$ up-steps along the vector $(1, 1)$ and $(n + \ell)$ down-steps along the vector $(1, -1)$. Since

$$\#\mathcal{L} = \binom{2n - k}{n - k - \ell, n + \ell} = \binom{2n - k}{n + \ell},$$

it is enough to construct a bijection Φ between \mathcal{V} and \mathcal{L} .

Before constructing the bijection we first introduce two well-known bijections φ and ψ between rooted ordered trees and Dyck paths.

The bijection φ corresponds a tree to a Dyck path by recording the steps when the tree is traversed in preorder. Here we record an up-step when we go down an edge and a down-step when going up. An example of the bijection φ is shown in the Figure 5.

The bijection ψ corresponds a tree to a Dyck path by recording the steps when the tree is traversed in preorder. Here, whenever we meet a vertex of

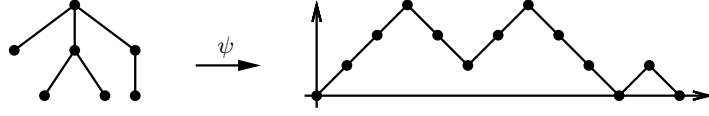
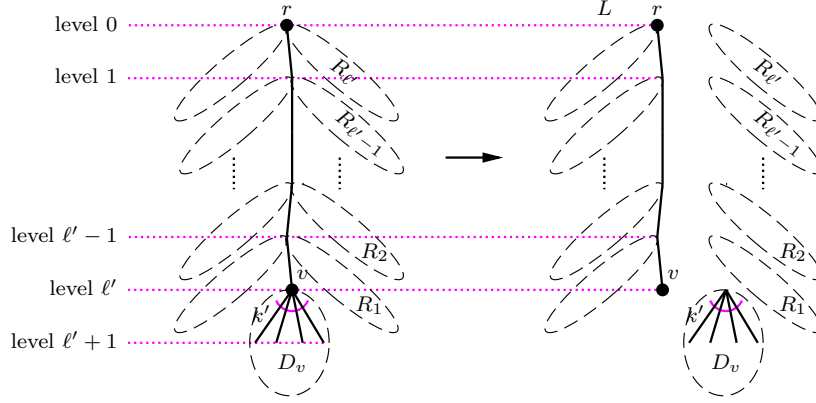

 FIGURE 6. The bijection ψ


FIGURE 7. Tree decomposition

outdegree k , except the last leaf, we record k up-steps and one down-step. An example of the bijection ψ is shown in the Figure 6.

Using two bijections φ and ψ , we will construct another bijection Φ between \mathcal{V} to \mathcal{L} : given $(T, v) \in \mathcal{V}$, let $k'(\geq k)$ be the number of children of v in T and let $\ell'(\geq \ell)$ be the level of v in T . We decompose the tree T into $\ell' + 2$ subtrees: the subtree D_v consisting of descendants of v , ℓ' subtrees $R_1, R_2, \dots, R_{\ell'}$ on the right hand side of the path from v to the root r of T , and the remaining tree L as illustrated in the Figure 7. Clearly, the outdegree of the root of D_v is k' . In preorder, the vertex v is the last leaf at level ℓ' in the tree L .

From (T, v) , we define a lattice path P of length $(2n + \ell' + 1)$ from $(0, 0)$ to $(2n + \ell' + 1, -\ell' - 1)$ by

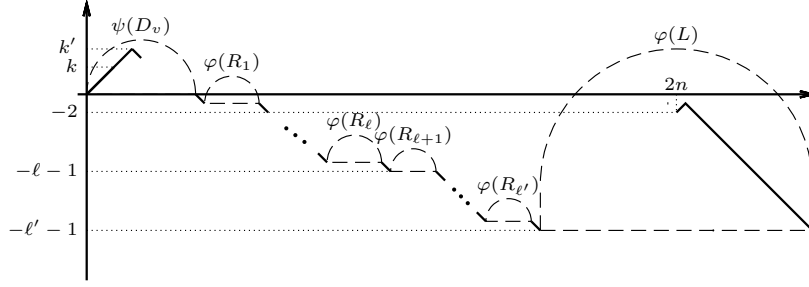
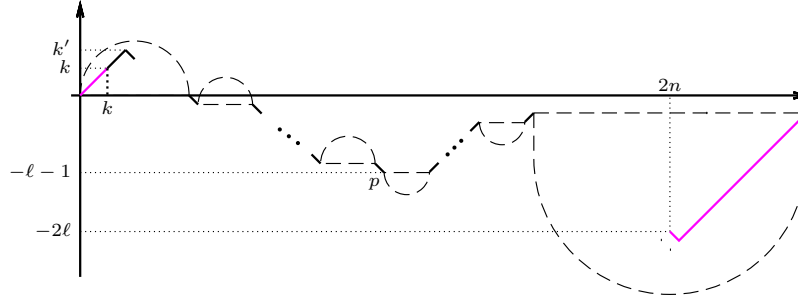
$$P = \psi(D_v) \searrow \varphi(R_1) \searrow \varphi(R_2) \searrow \cdots \searrow \varphi(R_{\ell'}) \searrow \varphi(L),$$

where \searrow means a down-step. By convention, if $\ell' = 0$. We set

$$P = \psi(D_v) \searrow \varphi(\emptyset).$$

Especially, if $\ell' > 0$, the lattice path P always starts with k' consecutive up-steps and ends with one up-step and ℓ' consecutive down-steps as the Figure 8.

When we draw the line $y = -\ell - 1$, because the y -coordinate of the lowest point of P is $-\ell' - 1$, this line should meet the lattice path P . Denote p by the first meeting point of P and $y = -\ell - 1$. Replacing the portion of P from p to $(2n + \ell' + 1, -\ell' - 1)$ by its reflection about $y = -\ell - 1$, we obtain the lattice path \tilde{P} from $(0, 0)$ to $(2n + \ell' + 1, -2\ell + \ell' - 1)$, which also always starts with k' consecutive up-steps and ends with one down-step and

FIGURE 8. Outline of a lattice path P induced from tree decompositionFIGURE 9. Outline of a lattice path \tilde{P} and \hat{P} by reflection about $y = -\ell - 1$

ℓ' consecutive up-steps as the Figure 9, even if $\ell' = 0$. Note that, if $\ell' = 0$, p should be the last point $(2n + 1, -1)$ of P and $\tilde{P} = P$.

By removing the first k steps and the last $(\ell' + 1)$ steps from \tilde{P} , we obtain the lattice path \hat{P} from (k, k) to $(2n, -2\ell)$. Thus \hat{P} belongs to \mathcal{L} and define $\Phi(T, v) = \hat{P}$. Since the reflection and the removal are reversible, Φ is a bijection. \square

Theorem 10. *Given $n \geq 1$, for any two nonnegative integers k and ℓ , the number of vertices of outdegree k at level ℓ among trees in \mathcal{T}_n is*

$$\frac{k + 2\ell}{2n - k} \binom{2n - k}{n + \ell}. \quad (3)$$

Proof. Using a sieve method, we obtain (3) from (2) by

$$\binom{2n - k}{n + \ell} - \binom{2n - k - 1}{n + \ell} - \binom{2n - k}{n + \ell + 1} + \binom{2n - k - 1}{n + \ell + 1}.$$

\square

Thus, obtaining (1) from (3) by changing the index ℓ , we may count the set \mathcal{D} as (iv) of Theorem 4.

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